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A unified algebraic approach to the classical Yang–Baxter equation

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Abstract
In this paper, the different operator forms of the classical Yang–Baxter equation are given in the tensor expression through a unified algebraic method. It is closely related to left-symmetric algebras which play an important role in many fields in mathematics and mathematical physics. By studying the relations between left-symmetric algebras and the classical Yang–Baxter equation, we can construct left-symmetric algebras from certain classical r-matrices and conversely, there is a natural classical r-matrix constructed from a left-symmetric algebra which corresponds to a parakähler structure in geometry. Moreover, the former in a special case gives an algebraic interpretation of the 'left symmetry' as a Lie bracket 'left-twisted' by a classical r-matrix.

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1. Introduction

The classical Yang–Baxter equation (CYBE) first arose in the study of inverse scattering theory [1, 2]. It is also a special case of the Schouten bracket in differential geometry, which was introduced in 1940 [3]. It can be regarded as a 'classical limit' of the quantum Yang–Baxter equation [4]. It plays a crucial role in many fields such as symplectic geometry, integrable systems, quantum groups, quantum field theory and so on (see [5] and the references therein). The Yang–Baxter system has become an important topic in both mathematics and mathematical physics since 1980s.

The standard form of the CYBE in a Lie algebra is given in the tensor expression as follows. Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \otimes \mathcal{G}$. r is called a solution of the CYBE in $\mathcal{G}$ if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{in} \quad U(\mathcal{G}), \tag{1.1}$$

where $U(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$ and for $r = \sum_i a_i \otimes b_i$,

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{1.2}$$

r is also called a classical r-matrix due to the expression of r under a basis of $\mathcal{G}$.

There are a lot of results on the CYBE when $\mathcal{G}$ is semisimple (cf [6, 7], etc). However, it is not easy to study equation (1.1) directly in a general case. A natural idea is to replace the tensor form by a linear operator. There are several approaches. In [8], Semonov–Tian–Shansky studied the CYBE systematically. In particular, an operator form of the CYBE is given as a linear map $R : \mathcal{G} \to \mathcal{G}$ satisfying

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \qquad \forall x, \quad y \in \mathcal{G}. \tag{1.3}$$

It is equivalent to the tensor form (1.1) of the CYBE when the following two conditions are satisfied: (a) there exists a nondegenerate symmetric invariant bilinear form on $\mathcal{G}$ and (b) r is skew-symmetric. However, the relation between the operator form (1.3) and the tensor form (1.1) in a general case is still not clear.

Later, Kupershmidt restudied the CYBE in [9]. When r is skew-symmetric, the tensor form (1.1) of the CYBE is equivalent to a linear map $r : \mathcal{G}^* \to \mathcal{G}$ satisfying

$$[r(x), r(y)] = r(\mathrm{ad}^* r(x)(y) - \mathrm{ad}^* r(y)(x)), \qquad \forall x, \quad y \in \mathcal{G}^*, \tag{1.4}$$

where $\mathcal{G}^*$ is the dual space of $\mathcal{G}$ and ad^* is the dual representation of adjoint representation (coadjoint representation) of the Lie algebra $\mathcal{G}$. Moreover, Kupershmidt generalized the above ad^* to be an arbitrary representation $\rho : \mathcal{G} \to gl(V)$ of $\mathcal{G}$, that is, a linear map $T : V \to \mathcal{G}$ satisfying

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \qquad \forall u, \quad v \in V, \tag{1.5}$$

which was regarded as a natural generalization of the CYBE. Such an operator is called an $\mathcal{O}$-operator associated with ρ. Note that the operator form (1.3) of the CYBE given by Semonov–Tian–Shansky is just an $\mathcal{O}$-operator associated with the adjoint representation of $\mathcal{G}$. However, there is no direct relation between the $\mathcal{O}$-operators and the tensor form (1.1) of the CYBE, either.

In this paper, we give a further study of the CYBE which unifies the above different operator forms of the CYBE. The key is how to interpret the $\mathcal{O}$-operators in terms of the tensor expression. Our idea is to extend the Lie algebra $\mathcal{G}$ to construct a bigger Lie algebra such that the $\mathcal{O}$-operators can be related to the solutions of the tensor form of the CYBE in it. Thus we can obtain not only the direct relations between the above operators and the tensor form (1.1) of the CYBE, but also the corresponding results of Semonov–Tian–Shansky and Kupershmidt as special cases. It is quite similar to the double construction [7].

Furthermore, there is an algebraic structure behind the above study. It is the left-symmetric algebra (or under other names such as pre-Lie algebra, quasi-associative algebra, Vinberg algebra and so on). Left-symmetric algebras are a class of nonassociative algebras coming from the study of convex homogeneous cones, affine manifolds and affine structures on Lie groups, deformation of associative algebras [10–13] and then appear in many fields in mathematics and mathematical physics, such as complex and symplectic structures on Lie groups and Lie algebras [14–18], integrable systems [19, 20], Poisson brackets and infinite-dimensional Lie algebras [21–23], vertex algebras [24], quantum field theory [25], operads [26] and so on (more examples can be found in a survey in [27] and the references therein).

Although some scattered results are known in certain references (cf [9, 28–31], etc), we give a systematic study on the relations between left-symmetric algebras and the CYBE in this paper. It can be regarded as a generalization of the correspondence between left-symmetric algebras and bijective 1-cocycles whose existence gives a necessary and sufficient condition for a Lie algebra with a compatible left-symmetric algebra structure. For the study of left-symmetric algebras, it provides a construction from certain classical r-matrices. In particular, as a special case, an algebraic interpretation of the so-called 'left-symmetry' is induced by equation (1.3): in a certain sense, the 'left symmetry' can be interpreted as a Lie bracket 'left-twisted' by a classical r-matrix. On the other hand, for the study of the CYBE, there is a natural classical r-matrix with a simple form constructed from a left-symmetric algebra which corresponds to a parakähler structure in geometry.

The paper is organized as follows. In section 2, we construct a direct relation between $\mathcal{O}$-operators and the tensor form of the CYBE. In the cases of adjoint representations and co-adjoint representations, we can get the operator forms (1.3) and (1.4) of the CYBE. In section 3, we briefly introduce left-symmetric algebras and then study the relations between them and the CYBE. In section 4, we summarize the main results obtained in the previous sections.

Throughout this paper, without special saying, all algebras are of finite dimension and over an algebraically closed field of characteristic 0 and r is a solution of the CYBE or r is a classical r-matrix refers to that r satisfies the tensor form (1.1) of the CYBE.

2. $\mathcal{O}$-operators and the tensor form of the CYBE

At first, we give some notations. Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \otimes \mathcal{G}$. r is said to be skew-symmetric if

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i). \tag{2.1}$$

For $r = \sum_i a_i \otimes b_i \in \mathcal{G} \otimes \mathcal{G}$, we denote

$$r^{21} = \sum_i b_i \otimes a_i. \tag{2.2}$$

On the other hand, let $\rho : \mathcal{G} \to gl(V)$ be a representation of the Lie algebra $\mathcal{G}$. On the vector space $\mathcal{G} \oplus V$, there is a natural Lie algebra structure (denoted by $\mathcal{G} \ltimes_\rho V$) given as follows [32]:

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1, \qquad \forall x_1, \quad x_2 \in \mathcal{G}, \qquad v_1, v_2 \in V. \tag{2.3}$$

Let $\rho^* : \mathcal{G} \to gl(V^*)$ be the dual representation of the representation $\rho : \mathcal{G} \to gl(V)$ of the Lie algebra $\mathcal{G}$. Then there is a close relation between the $\mathcal{O}$-operator associated with ρ and the (skew-symmetric) solutions of the CYBE in $\mathcal{G} \ltimes_{\rho^*} V^*$.

Any linear map $T : V \to \mathcal{G}$ can be identified as an element in $\mathcal{G} \otimes V^* \subset (\mathcal{G} \ltimes_{\rho^*} V^*) \otimes (\mathcal{G} \ltimes_{\rho^*} V^*)$ as follows. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{G}$. Let $\{v_1, \ldots, v_m\}$ be a basis of V and $\{v_1^*, \ldots, v_m^*\}$ be its dual basis, that is, $v_i^*(v_j) = \delta_{ij}$. Set $T(v_i) = \sum_{j=1}^n a_{ij} e_j, i = 1, \ldots, m$. Since as vector spaces, $\mathrm{Hom}(V, \mathcal{G}) \cong \mathcal{G} \otimes V^*$, we have

$$T = \sum_{i=1}^m T(v_i) \otimes v_i^* = \sum_{i=1}^m \sum_{j=1}^n a_{ij} e_j \otimes v_i^* \in \mathcal{G} \otimes V^* \subset (\mathcal{G} \ltimes_{\rho^*} V^*) \otimes (\mathcal{G} \ltimes_{\rho^*} V^*). \tag{2.4}$$

Claim. *$r = T - T^{21}$ is a skew-symmetric solution of the CYBE in $\mathcal{G} \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator.*

In fact, by equation (2.4), we have

$$[r_{12}, r_{13}] = \sum_{i,k=1}^{m} \{[T(v_i), T(v_k)] \otimes v_i^* \otimes v_k^* - \rho^*(T(v_i))v_k^* \otimes v_i^* \otimes T(v_k) + \rho^*(T(v_k))v_i^* \otimes T(v_i) \otimes v_k^*\},$$

$$[r_{12}, r_{23}] = \sum_{i,k=1}^{m} \{-v_i^* \otimes [T(v_i), T(v_k)] \otimes v_k^* - T(v_i) \otimes \rho^*(T(v_k))v_i^* \otimes v_k^* + u_i^* \otimes \rho^*(T(v_i))v_k^* \otimes T(v_k)\},$$

$$[r_{13}, r_{23}] = \sum_{i,k=1}^{m} \{v_i^* \otimes u_k^* \otimes [T(v_i), T(v_k)] + T(v_i) \otimes v_k^* \otimes \rho^*(T(v_k))v_i^* - v_i^* \otimes T(v_k) \otimes \rho^*(T(v_i))v_k^*\}.$$

By the definition of dual representation, we know

$$\rho^*(T(v_k))v_i^* = -\sum_{j=1}^{m} v_i^*(\rho(T(v_k))v_j)v_j^*.$$

Thus,

$$-\sum_{i,k=1}^{m} T(v_i) \otimes \rho^*(T(v_k))v_i^* \otimes v_k^* = -\sum_{i,k=1}^{m} T(v_i) \otimes \left[\sum_{j=1}^{m} -v_i^*(\rho(T(v_k))v_j)v_j^*\right] \otimes v_k^*$$
$$= \sum_{i,k=1}^{m}\sum_{j=1}^{m} v_j^*(\rho(T(v_k))v_i)T(v_j) \otimes v_i^* \otimes v_k^* = \sum_{i,k=1}^{m} T\left(\sum_{j=1}^{m}(v_j^*(\rho(T(v_k))v_i)v_j\right) \otimes v_i^* \otimes v_k^*$$
$$= \sum_{i,k=1}^{m} T(\rho(T(v_k))v_i) \otimes v_i^* \otimes v_k^*.$$

Therefore,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$
$$= \sum_{i,k=1}^{m} \{([T(v_i, v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k)) \otimes v_i^* \otimes v_k^* - v_i^* \otimes ([T(v_i, v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k)) \otimes v_k^* + v_i^* \otimes v_k^* \otimes ([T(v_i, v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k))\}.$$

So r is a classical r-matrix in $\mathcal{G} \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator.

Obviously, the above r is exactly the skew-symmetric classical r-matrix in $\mathcal{G} \ltimes_{\rho^*} V^*$ which is in $\mathcal{G} \otimes V^* - V^* \otimes \mathcal{G}$.

Next we consider the cases that ρ is the adjoint representation ad : $\mathcal{G} \to gl(\mathcal{G})$ or the coadjoint representation $\mathrm{ad}^* : \mathcal{G} \to gl(\mathcal{G}^*)$ with $\langle \mathrm{ad}^* x(y^*), z\rangle = -\langle y^*, [x, z]\rangle$ for any $x, z \in \mathcal{G}$ and $y^* \in \mathcal{G}^*$, where $\langle , \rangle$ is the ordinary pair between $\mathcal{G}$ and $\mathcal{G}^*$.

Case I. $\rho = \mathrm{ad}^*$, the coadjoint representation. In this case, $V = \mathcal{G}^*$ and $V^* = \mathcal{G}$. For any linear map $T : \mathcal{G}^* \to \mathcal{G}$, T can be identified as an element in $\mathcal{G} \otimes \mathcal{G}$ by

$$\langle T(u), v\rangle = \langle u \otimes v, T\rangle, \qquad \forall u, \quad v \in \mathcal{G}^*. \tag{2.5}$$

Therefore, although $r = T - T^{21} \in (\mathcal{G} \ltimes_{\mathrm{ad}} \mathcal{G}) \otimes (\mathcal{G} \ltimes_{\mathrm{ad}} \mathcal{G})$, in fact, $r \in \mathcal{G} \otimes \mathcal{G}$. Hence, r is a skew-symmetric solution of the CYBE in $\mathcal{G}$ if and only if T satisfies equation (1.4).

In particular, suppose that $r \in \mathcal{G} \otimes \mathcal{G}$ which is identified as a linear map from $\mathcal{G}^*$ to $\mathcal{G}$ and r itself is skew-symmetric. Then $r - r^{21} = 2r$. Obviously, r is a classical r-matrix if and only if $2r$ is also a classical r-matrix. Therefore r is a solution of the CYBE in $\mathcal{G}$ if and only if r satisfies equation (1.4). Thus, in this case, the CYBE in the tensor expression (1.1) is equivalent to equation (1.4), which was given by Kupershmidt [9].

Case II. $\rho = \mathrm{ad}$, the adjoint representation. In this case, we suppose that the Lie algebra $\mathcal{G}$ is equipped with a nondegenerate invariant symmetric bilinear form $B(,)$. That is,

$$B(x, y) = B(y, x), \qquad B([x, y], z) = B(x, [y, z]), \qquad \forall x,\ y, z \in \mathcal{G}. \tag{2.6}$$

Hence, $\mathcal{G}^*$ is identified with $\mathcal{G}$. Let $r \in \mathcal{G} \otimes \mathcal{G}$. Then r can be identified as a linear map from $\mathcal{G}$ to $\mathcal{G}$. If r is skew-symmetric, then r is a solution of the CYBE in $\mathcal{G}$ if and only if r satisfies equation (1.3). Thus, in this case, the CYBE in the tensor expression (1.1) is equivalent to equation (1.3), which was given by Semonov–Tian–Shansky [8].

3. The CYBE and left-symmetric algebras

A left-symmetric algebra A is a vector space over a field $\mathbf{F}$ equipped with a bilinear product $(x, y) \to xy$ satisfying that for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \tag{3.1}$$

is symmetric in x, y, that is,

$$(x, y, z) = (y, x, z), \qquad \text{or equivalently} \qquad (xy)z - x(yz) = (yx)z - y(xz). \tag{3.2}$$

Left-symmetric algebras are Lie-admissible algebras (cf [33, 34]). In fact, let A be a left-symmetric algebra. Then the commutator

$$[x, y] = xy - yx, \qquad \forall x, \quad y \in A, \tag{3.3}$$

defines a Lie algebra $\mathcal{G}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called the compatible left-symmetric algebra structure on the Lie algebra $\mathcal{G}(A)$.

Furthermore, for any $x \in A$, let L_x denote the left multiplication operator, that is, $L_x(y) = xy$ for any $y \in A$. Then $L : \mathcal{G}(A) \to gl(\mathcal{G}(A))$ with $x \to L_x$ gives a regular representation of the Lie algebra $\mathcal{G}(A)$, that is,

$$[L_x, L_y] = L_{[x,y]}, \qquad \forall x, \quad y \in A. \tag{3.4}$$

It is not true that there is a compatible left-symmetric algebra structure on every Lie algebra. For example, a real or complex Lie algebra $\mathcal{G}$ with a compatible left-symmetric algebra structure must satisfy the condition $[\mathcal{G}, \mathcal{G}] \neq \mathcal{G}$ [34]; hence, there does not exist a compatible left-symmetric algebra structure on any real or complex semisimple Lie algebra. Here, we briefly introduce a necessary and sufficient condition for a Lie algebra with a compatible left-symmetric algebra structure [33]. Let $\mathcal{G}$ be a Lie algebra and $\rho : \mathcal{G} \to gl(V)$ be a representation of $\mathcal{G}$. A 1-cocycle q associated with ρ (denoted by (ρ, q)) is defined as a linear map from $\mathcal{G}$ to V, satisfying

$$q[x, y] = \rho(x)q(y) - \rho(y)q(x), \qquad \forall x, \quad y \in \mathcal{G}. \tag{3.5}$$

Then there is a compatible left-symmetric algebra structure on $\mathcal{G}$ if and only there exists a bijective 1-cocycle of $\mathcal{G}$. In fact, let (ρ, q) be a bijective 1-cocycle of $\mathcal{G}$. Then

$$x * y = q^{-1}\rho(x)q(y), \qquad \forall x, \quad y \in \mathcal{G}, \tag{3.6}$$

defines a compatible left-symmetric algebra structure on $\mathcal{G}$. Conversely, for a left-symmetric algebra A, (L, id) is a bijective 1-cocycle of $\mathcal{G}(A)$, where id is the identity transformation on $\mathcal{G}(A)$.

Note that for any Lie algebra $\mathcal{G}$ and its representation $\rho : \mathcal{G} \to gl(V)$, a linear isomorphism $T : V \to \mathcal{G}$ (hence $\dim \mathcal{G} = \dim V$) is an $\mathcal{O}$-operator associated with ρ if and only if T^{-1} is a (bijective) 1-cocycle of $\mathcal{G}$ associated with ρ. Therefore, if $q : \mathcal{G} \to V$ is a bijective 1-cocycle of $\mathcal{G}$ associated with ρ, then $q^{-1} - (q^{-1})^{21}$ is a solution of the CYBE in $\mathcal{G} \ltimes_{\rho^*} V^*$. In particular, let $(\mathcal{G}, *)$ be a left-symmetric algebra. Since $T = id$ is an $\mathcal{O}$-operator associated with the regular representation L, we have

$$r = \sum_{i}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{3.7}$$

is a solution of the CYBE in $\mathcal{G} \ltimes_{L^*} \mathcal{G}^*$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathcal{G}$ and $\{e_1^*, \ldots, e_n^*\}$ is its dual basis. Moreover, we would like to point out that in the Lie algebra $\mathcal{G}(A) \ltimes_{\mathrm{ad}^*} \mathcal{G}^*(A)$, equation (3.7) is not a solution of the CYBE, but a solution of the so-called modified CYBE [8]. That is, in $\mathcal{G}(A) \ltimes_{\mathrm{ad}^*} \mathcal{G}^*(A)$, equation (3.7) does not satisfy equation (1.1), but satisfies

$$[x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]] = 0, \quad \forall\, x \in \mathcal{G}(A) \ltimes_{\mathrm{ad}^*} \mathcal{G}^*(A). \tag{3.8}$$

On the other hand, if an $\mathcal{O}$-operator T associated with ρ is invertible, then T^{-1} is a bijective 1-cocycle of $\mathcal{G}$ associated with ρ. Hence,

$$x \cdot y = T(\rho(x)(T^{-1}(y))), \qquad \forall\, x, \quad y \in \mathcal{G}, \tag{3.9}$$

defines a compatible left-symmetric algebra structure on $\mathcal{G}$ through equation (3.6) by letting $q = T^{-1}$. Moreover, for any $u, v \in V$, let $x = T(u)$, $y = T(v)$. Thus by equation (3.9), we have

$$T(u) \cdot T(v) = T(\rho(T(u))v).$$

Since T is invertible, there exists a left-symmetric algebra structure on V induced from the left-symmetric algebra structure on $\mathcal{G}$ by

$$u * v = T^{-1}(T(u) \cdot T(v)) = \rho(T(u))(v), \qquad \forall\, u, \quad v \in V. \tag{3.10}$$

It is obvious that T is an isomorphism of left-symmetric algebras between them.

Furthermore, we can generalize the above construction of left-symmetric algebras to a general $\mathcal{O}$-operator. Let $\mathcal{G}$ be a Lie algebra and $\rho : \mathcal{G} \to gl(V)$ be its representation. Let $T : V \to \mathcal{G}$ be a linear map. Then on V, the new product

$$u * v = \rho(T(u))v, \qquad \forall\, u, \quad v \in V \tag{3.11}$$

satisfies that for any $u, v, w \in V$,

$$\begin{aligned}(u, v, w) - (v, u, w) &= \rho(T\rho(T(u))v)w - \rho(T(u))\rho(T(v))w \\ &\quad - \rho(T\rho(T(v))u)w + \rho(T(v))\rho(T(u))w \\ &= \rho([T(v), T(u)])w + \rho(T(\rho(T(u))v - \rho(T(v))u))w.\end{aligned}$$

Hence, equation (3.11) defines a left-symmetric algebra if and only if

$$[T(u), T(v)] - T(\rho(T(u))v - \rho(T(v))u) \in \operatorname{Ker}\rho, \qquad \forall\, u, \quad v \in V, \tag{3.12}$$

where $\operatorname{Ker}\rho = \{x \in \mathcal{G} | \rho(x) = 0\}$. In particular, for any $\mathcal{O}$-operator $T : V \to \mathcal{G}$ associated with ρ, equation (3.11) defines a left-symmetric algebra on V. Therefore, V is a Lie algebra as the sub-adjacent Lie algebra of this left-symmetric algebra and T is a Lie algebraic homomorphism. Furthermore, $T(V) = \{T(v) | v \in V\} \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$ and there is an induced left-symmetric algebra structure on $T(V)$ given by

$$T(u) \cdot T(v) = T(u * v), \qquad \forall\, u, \quad v \in V. \tag{3.13}$$

Moreover, its sub-adjacent Lie algebra structure is just the Lie subalgebra structure of $\mathcal{G}$ and T is a homomorphism of left-symmetric algebras.

In fact, the above Lie algebra structure on V coincides the standard construction of Lie bialgebra from the classical r-matrix $r = T - T^{21}$ as follows. $\mathcal{G} \ltimes_{\rho^*} V^*$ is a Lie bialgebra [5, 7] with the cobracket $\delta(f) = [f \otimes 1 + 1 \otimes f, r]$ for any $f \in \mathcal{G} \ltimes_{\rho^*} V^*$. Hence, V^* is a Lie co-subalgebra of $\mathcal{G} \ltimes_{\rho^*} V^*$. Therefore, V is a Lie algebra just given by $[u, v] = \rho(T(u))v - \rho(T(v))u$ for any $u, v \in V$.

According to Drinfel'd [7], $r \in \mathcal{G} \otimes \mathcal{G}$ is a skew-symmetric and nondegenerate solution of the CYBE in $\mathcal{G}$ if and only if the bilinear form B on $\mathcal{G}$ given by

$$B(x, y) = \langle r^{-1}(x), y\rangle, \qquad \forall x, \quad y \in \mathcal{G}, \tag{3.14}$$

is a 2-cocycle on $\mathcal{G}$, that is,

$$B([x, y], z) + B([y, z], x) + B([z, x], y) = 0, \qquad \forall x, \quad y, z \in \mathcal{G}. \tag{3.15}$$

In geometry, a skew-symmetric and nondegenerate 2-cocycle on a Lie algebra $\mathcal{G}$ is also called a symplectic form which corresponds to a symplectic form on a Lie group whose Lie algebra is $\mathcal{G}$, and such a Lie group (or Lie algebra) is called a symplectic Lie group (or Lie algebra) [15].

Now, let us return to our study on left-symmetric algebras and the CYBE. Let A be a left-symmetric algebra. Then the classical r-matrix r given by equation (3.7) is nondegenerate. Moreover, $r : (\mathcal{G} \ltimes_{L^*} \mathcal{G}^*)^* \to \mathcal{G} \ltimes_{L^*} \mathcal{G}^*$ satisfies the following equations:

$$r(e_i^*) = e_i^*, \qquad r(e_i) = -e_i, \qquad i = 1, \ldots, n.$$

Therefore, for any i, j, k, l, we have

$$\langle r^{-1}(e_i + e_j^*), e_k + e_l^*\rangle = \langle -e_i + e_j^*, e_k + e_l^*\rangle = \langle -e_i, e_l^*\rangle + \langle e_k, e_j^*\rangle.$$

Hence, there is a natural 2-cocycle ω (symplectic form) on $\mathcal{G} \ltimes_{L^*} \mathcal{G}^*$ induced by r^{-1} : $\mathcal{G} \ltimes_{L^*} \mathcal{G}^* \to (\mathcal{G} \ltimes_{L^*} \mathcal{G}^*)^*$ given by

$$\omega(x + x^*, y + y^*) = \langle x^*, y\rangle - \langle y^*, x\rangle, \qquad \forall x, \ y \in \mathcal{G}, \quad x^*, \ y^* \in \mathcal{G}^*. \tag{3.16}$$

The above structure $\mathcal{G} \ltimes_{L^*} \mathcal{G}^*$ with the symplectic form ω (3.16) corresponds to a parakähler structure. In geometry, a parakähler manifold is a symplectic manifold with a pair of transversal Lagrangian foliations [35]. A parakähler Lie algebra $\mathcal{G}$ is just the Lie algebra of a Lie group G with a G-invariant parakähler structure [36]. On the other hand, such a structure is just a phase space of $\mathcal{G}$ in mathematical physics [37–39].

Next we still consider the case that $\rho = \mathrm{ad}$, the adjoint representation. Let $\mathcal{G}$ be a Lie algebra and f be a linear transformation on $\mathcal{G}$. Then on $\mathcal{G}$, the new product

$$x * y = [f(x), y], \qquad \forall x, \ y \in \mathcal{G}, \tag{3.17}$$

defines a left-symmetric algebra if and only if

$$[f(x), f(y)] - f([f(x), y] + [x, f(y)]) \in C(\mathcal{G}), \qquad \forall x, \ y \in \mathcal{G}, \tag{3.18}$$

where $C(\mathcal{G}) = \{x \in \mathcal{G} | [x, y] = 0, \forall y \in \mathcal{G}\}$ is the center of Lie algebra $\mathcal{G}$. In particular, the map given by equation (1.3) defines a left-symmetric algebra on $\mathcal{G}$ through equation (3.17). On the other hand, if in addition, the center $C(\mathcal{G})$ is zero, then the linear transformation f satisfying equation (3.18) just satisfies equation (1.3), that is, f satisfies the operator form of the CYBE.

The formula (3.17) was also given in [31], and a similar construction for Novikov algebras (left-symmetric algebras with commutative right multiplication operators) was given with r satisfying some additional conditions in [40]. We would like to point out that the above construction cannot get all left-symmetric algebras.

Moreover, the above discussion gives an algebraic interpretation of 'left-symmetry' (3.2). Let $\{e_i\}$ be a basis of a Lie algebra $\mathcal{G}$ and r be a linear transformation satisfying equation (1.3). Set $r(e_i) = \sum_{j\in I} r_{ij} e_j$. Then the basis interpretation of equation (3.17) is given as

$$e_i * e_j = \sum_{l\in I} r_{il}[e_l, e_j]. \tag{3.19}$$

In this sense, such a construction of left-symmetric algebras can be regarded as a Lie algebra 'left-twisted' by a classical r-matrix. On the other hand, we consider the right-symmetric algebra, that is, $(x, y, z) = (x, z, y)$ for any $x, y, z \in A$, where (x, y, z) is the associator given by equation (3.1). Set

$$e_i \cdot e_j = [e_i, r(e_j)] = \sum_{l\in I} r_{jl}[e_i, e_l]. \tag{3.20}$$

Then the above product defines a right-symmetric algebra on $\mathcal{G}$, which can be regarded as a Lie algebra 'right-twisted' by a classical r-matrix.

Roughly speaking, the CYBE describes certain 'permutation' relation and left-symmetry or right-symmetry is a kind of special 'permutation'. There is a close relation between them by equation (3.19) or (3.20).

At the end of this section, we give a further study on the linear transformations satisfying equation (3.18). For a left-symmetric algebra structure on $\mathcal{G}$ given by equation (3.17) with f satisfying equation (3.18), if in addition, its sub-adjacent Lie algebra is just $\mathcal{G}$ itself, that is,

$$[x, y] = [f(x), y] - [y, f(x)], \qquad \forall x, y \in \mathcal{G}, \tag{3.21}$$

then it is a left-symmetric inner derivation algebra, that is, for every $x \in \mathcal{G}$, L_x is an interior derivation of the Lie algebra $\mathcal{G}$. Such a structure corresponds to a flat left-invariant connection adapted with the interior automorphism structure of a Lie group, which was first studied in [34]. Moreover, every left-symmetric inner derivation algebra can be obtained by this way. On the other hand, if f satisfies equation (1.3) and f is invertible, then f satisfies equation (3.21) if and only if f is an automorphism of $\mathcal{G}$. Under this condition, $\mathcal{G}$ must be solvable since the sub-adjacent Lie algebra of a left-symmetric inner derivation algebra is solvable [34].

Moreover, there is a general conclusion. Let $\mathcal{G}$ be a complex Lie algebra with a nondegenerate symmetric invariant bilinear form. If there exists an invertible skew-symmetric classical r-matrix r, then r can be identified as a linear transformation on $\mathcal{G}$ satisfying equation (1.3). Hence, r^{-1} is a derivation of $\mathcal{G}$. Since a complex Lie algebra with a nondegenerate derivation must be nilpotent (cf [41]), $\mathcal{G}$ is nilpotent. This conclusion also generalizes a similar result for complex semisimple Lie algebras ([5], Proposition 2.2.5).

4. Summary

In this paper, we interpret the $\mathcal{O}$-operators in terms of the tensor expression. Thus, the different operator forms of the CYBE are given in the tensor expression through a unified algebraic method. Since the Lie bialgebra structures are obtained through the solutions of the CYBE in the tensor form as

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \qquad \forall x \in \mathcal{G},$$

it is easy to get the corresponding Lie bialgebra structures from the different operator forms of the CYBE through our study. It will also be useful to consider the quantization of these Lie bialgebra structures (for example, try to find the corresponding Drinfel'd quantum twist [5, 7]).

On the other hand, there are close relations between left-symmetric algebras and the CYBE. They can be regarded as a generalization of the study of $\mathcal{O}$-operators in the cases that the $\mathcal{O}$-operators are invertible. We can construct left-symmetric algebras from certain classical r-matrices, and conversely, there is a natural classical r-matrix constructed from a left-symmetric algebra which corresponds to a parakähler structure in geometry. Moreover, the former in a special case gives an algebraic interpretation of the 'left symmetry' as a Lie bracket 'left-twisted' by a classical r-matrix.

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